Testing the Monogamy Relations via Rank-2 Mixtures

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Abstract

We introduce two tangle-based four-party entanglement measures t_1 and t_2, and two negativity-based measures n_1 and n_2, which are derived from the monogamy relations. These measures are computed for three four-qubit maximally entangled and W states explicitly. We also compute these measures for the rank-2 mixture $\rho_4 = p|\text{GHZ}_4\rangle\langle\text{GHZ}_4| + (1-p)|\text{W}_4\rangle\langle\text{W}_4|$ by finding the corresponding optimal decompositions. It turns out that $t_1(\rho_4)$ is trivial and the corresponding optimal decomposition is equal to the spectral decomposition. Probably, this triviality is a sign of the fact that the corresponding monogamy inequality is not sufficiently tight. We fail to compute $t_2(\rho_4)$ due to the difficulty for the calculation of the residual entanglement. The negativity-based measures $n_1(\rho_4)$ and $n_2(\rho_4)$ are explicitly computed and the corresponding optimal decompositions are also derived explicitly.

I. INTRODUCTION

Research into entanglement of quantum states has long history from the very beginning of quantum mechanics[1, 2]. At that time the main motivation for the study of entanglement was pure theoretical. It was to explore the non-local properties of quantum mechanics. Recent considerable attention to the quantum entanglement[3, 4] has both theoretical and practical aspects. While the former is for understanding of quantum information theories more deeply, the latter is for developing the quantum technology. As shown for last two decades quantum entanglement plays a central role in quantum teleportation[5], superdense coding[6], quantum cloning[7], and quantum cryptography[8, 9]. It is also quantum entanglement, which makes the quantum computer[1] outperform the classical one[11]. Thus, it is very important to understand how to quantify and how to characterize the entanglement. Still, however, this issue is not completely understood.

For bipartite quantum system many entanglement measures were constructed before such as distillable entanglement[12], entanglement of formation (EOF)[12], and relative entropy of entanglement (REE)[13, 14]. Among them[2] the closed formula for the analytic computation of EOF were found in Ref. [16] via the concurrence $\mathcal{C}$ as

$$E_F(\mathcal{C}) = h\left(\frac{1 + \sqrt{1 - \mathcal{C}^2}}{2}\right),\tag{1.1}$$

where $h(x)$ is a binary entropy function $h(x) = -x \ln x - (1 - x)\ln(1 - x)$. For two-qubit pure state $|\psi\rangle_{AB} = \psi_{ij}|ij\rangle_{AB}$ with $(i, j = 0, 1)$, the concurrence $\mathcal{C}_{A|B}$ between party A and party B is given by

$$\mathcal{C}_{A|B} = |\epsilon_{i_1 i_2}\epsilon_{j_1 j_2}\psi_{i_1 j_1}\psi_{i_2 j_2}| = 2|\psi_{00}\psi_{11} - \psi_{01}\psi_{10}|,\tag{1.2}$$

where the Einstein convention is understood and $\epsilon_{\mu\nu}$ is an antisymmetric tensor. For two-qubit mixed state ρ_{AB} the concurrence $\mathcal{C}_{A|B}(\rho)$ can be computed by $\mathcal{C}_{A|B} = \max(\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0)$, where $\{\lambda_1^2, \lambda_2^2, \lambda_3^2, \lambda_4^2\}$ are eigenvalues of positive operator $\rho(\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$ with decreasing order. Thus, one can compute the EOF for all two-qubit states in principle.

Generalization to the multipartite entanglement is highly important and challenging issue in the context of quantum information theories. A seminal step toward this goal was initiated

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[1] The current status of quantum computer technology was reviewed in Ref.[10].
[2] Although there are a lot of attempts to derive the closed formula for REE, still we do not know how to compute the REE for the arbitrary two qubit mixtures except rare cases [15].

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in Ref. [17] by examining the three-qubit pure states. Authors in Ref. [17] have shown analytically the monogamy relation

$$\mathcal{C}^2_{q_1|(q_2 q_3)} \geq \mathcal{C}^2_{q_1|q_2} + \mathcal{C}^2_{q_1|q_3}. \tag{1.3}$$

This relation implies that the entanglement (measured by the squared concurrence) between q_1 and the remaining parties always exceeds entanglement between q_1 and q_2 plus entanglement between q_1 and q_3. This means that if q_1 and q_2 is maximally entangled, the whole system cannot have the tripartite entanglement. The inequality (1.3) is strong in a sense that the three-qubit W-state[18]

$$|\mathrm{W}_3\rangle = \frac{1}{\sqrt{3}}\left(|001\rangle + |010\rangle + |100\rangle\right) \tag{1.4}$$

saturates the inequality. Moreover, for three-qubit pure state $|\psi\rangle_{ABC} = \psi_{ijk}|ijk\rangle_{ABC}$ the leftover in the inequality

$$\tau_{A|B|C} = \mathcal{C}^2_{A|(BC)} - \left(\mathcal{C}^2_{A|B} + \mathcal{C}^2_{A|C}\right), \tag{1.5}$$

which we will call the residual entanglement[3], has following two expressions:

$$\tau_{A|B|C} = \left|2\epsilon_{i_1 i_2}\epsilon_{i_3 i_4}\epsilon_{j_1 j_2}\epsilon_{j_3 j_4}\epsilon_{k_1 k_3}\epsilon_{k_2 k_4}\psi_{i_1 j_1 k_1}\psi_{i_2 j_2 k_2}\psi_{i_3 j_3 k_3}\psi_{i_4 j_4 k_4}\right| = 4|d_1 - 2d_2 + 4d_3| \tag{1.6}$$

where

$$d_1 = \psi^2_{000}\psi^2_{111} + \psi^2_{001}\psi^2_{110} + \psi^2_{010}\psi^2_{101} + \psi^2_{100}\psi^2_{011}, \tag{1.7}$$

$$d_2 = \psi_{000}\psi_{111}\psi_{011}\psi_{100} + \psi_{000}\psi_{111}\psi_{101}\psi_{010} + \psi_{000}\psi_{111}\psi_{110}\psi_{001}$$

$$+ \psi_{011}\psi_{100}\psi_{101}\psi_{010} + \psi_{011}\psi_{100}\psi_{110}\psi_{001} + \psi_{101}\psi_{010}\psi_{110}\psi_{001},$$

$$d_3 = \psi_{000}\psi_{110}\psi_{101}\psi_{011} + \psi_{111}\psi_{001}\psi_{010}\psi_{100}.$$

From first expression one can show that $\tau_{A|B|C}$ is invariant under a stochastic local operation and classical communication (SLOCC)[19]. From second expression one can show that $\tau_{A|B|C}$ is invariant under the qubit permutation. It was also shown in Ref. [17] that $\tau_{A|B|C}$ is an entanglement monotone. Thus, the residual entanglement (or three-tangle) can play a role as an important measure for the genuine three-way entanglement.

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[3] In this paper $\sqrt{\tau_{A|B|C}}$ is called the three-tangle.

By making use of Eq. (1.6) one can compute the residual entanglement of all three-qubit pure states. For mixed state the residual entanglement is usually defined as a convex roof method[12, 20]

$$\tau_{A|B|C}(\rho) = \min \sum_i p_i \tau_{A|B|C}(|\psi_i\rangle\langle\psi_i|) \tag{1.8}$$

where the minimum is taken over all possible ensembles of pure states. The ensemble corresponding to the minimum of $\tau_{A|B|C}$ is called optimal decomposition. For given three-qubit mixed state it is highly difficult, in general, to find its optimal decomposition except very rare cases[21][4].

In order to find the entanglement measures in the multipartite system, there are two different approaches. First approach is to find the invariant monotones under the SLOCC transformation. As Ref.[24] has shown, any linearly homogeneous positive function of a pure state that is invariant under determinant 1 SLOCC operations is an entanglement monotone. Thus, the concurrence $\mathcal{C}_{A|B}$ and the three-tangle $\sqrt{\tau_{A|B|C}}$ are monotones. It is also possible to construct the SLOCC-invariant monotones in the higher-qubit systems. In the higher-qubit systems, however, there are many independent monotones, because the number of independent SLOCC-invariant monotones is equal to the degrees of freedom of pure quantum state minus the degrees of freedom induced by the determinant 1 SLOCC operations. For example, there are $2(2^n - 1) - 6n$ independent monotones in n-qubit system. Thus, in four-qubit system there are six invariant monotones. Among them, it was shown in Ref. [25] by making use of the antilinearity[20] that there are following three independent monotones which measure the true four-way entanglement:

$$
\begin{aligned}
\mathcal{F}_1^{(4)} &= (\sigma_\mu \sigma_\nu \sigma_2 \sigma_2) \bullet (\sigma^\mu \sigma_2 \sigma_\lambda \sigma_2) \bullet (\sigma_2 \sigma^\nu \sigma^\lambda \sigma_2) \\
\mathcal{F}_2^{(4)} &= (\sigma_\mu \sigma_\nu \sigma_2 \sigma_2) \bullet (\sigma^\mu \sigma_2 \sigma_\lambda \sigma_2) \bullet (\sigma_2 \sigma^\nu \sigma_2 \sigma_\tau) \bullet (\sigma_2 \sigma_2 \sigma^\lambda \sigma^\tau) \\
\mathcal{F}_3^{(4)} &= \frac{1}{2}(\sigma_\mu \sigma_\nu \sigma_2 \sigma_2) \bullet (\sigma^\mu \sigma^\nu \sigma_2 \sigma_2) \bullet (\sigma_\rho \sigma_2 \sigma_\tau \sigma_2) \bullet (\sigma^\rho \sigma_2 \sigma^\tau \sigma_2) \bullet (\sigma_\kappa \sigma_2 \sigma_2 \sigma_\lambda) \bullet (\sigma^\kappa \sigma_2 \sigma_2 \sigma^\lambda),
\end{aligned}
\tag{1.9}
$$

where $\sigma_0 = \mathbb{1}_2$, $\sigma_1 = \sigma_x$, $\sigma_2 = \sigma_y$, $\sigma_3 = \sigma_z$, and the Einstein convention is introduced with a metric $g^{\mu\nu} = \text{diag}\{-1, 1, 0, 1\}$. Furthermore, it was shown in Ref. [26] that there are

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[4] Recently, the three-tangle of the GHZ-symmetric states[22] has been computed analytically[23].

following three maximally entangled states in four-qubit system:

$$|\text{GHZ}_4\rangle = \frac{1}{\sqrt{2}}\big(|0000\rangle + |1111\rangle\big)$$

$$|\Phi_2\rangle = \frac{1}{\sqrt{6}}\Big(\sqrt{2}|1111\rangle + |1000\rangle + |0100\rangle + |0010\rangle + |0001\rangle\Big) \qquad (1.10)$$

$$|\Phi_3\rangle = \frac{1}{2}\big(|1111\rangle + |1100\rangle + |0010\rangle + |0001\rangle\big).$$

The measures $\mathcal{F}_1^{(4)}$, $\mathcal{F}_2^{(4)}$, and $\mathcal{F}_3^{(4)}$ of $|\text{GHZ}_4\rangle$, $|\Phi_2\rangle$, $|\Phi_3\rangle$, and

$$|\tilde{\text{W}}_4\rangle = \frac{1}{2}\big(|0111\rangle + |1011\rangle + |1101\rangle + |1110\rangle\big) \qquad (1.11)$$

are summarized in Table I. Recently, $\mathcal{F}_j^{(4)}$ $(j = 1,2,3)$ and the corresponding linear mono-tones $\mathcal{G}_j^{(4)}$ $(j = 1,2,3)$ for the rank-2 mixtures consist of one of the maximally entangled state and $|\tilde{\text{W}}_4\rangle$ are explicitly computed[27].

	$\mathcal{F}_1^{(4)}$	$\mathcal{F}_2^{(4)}$	$\mathcal{F}_3^{(4)}$
$|\Phi_1\rangle$	1	1	$\frac{1}{2}$
$|\Phi_2\rangle$	$\frac{8}{9}$	0	0
$|\Phi_3\rangle$	0	0	1
$|\tilde{\text{W}}_4\rangle$	0	0	0

Table I:$\mathcal{F}_1^{(4)}$, $\mathcal{F}_2^{(4)}$, and $\mathcal{F}_3^{(4)}$ of the maximally entangled and $\tilde{\text{W}}_4$ states.

Second approach is to find the monogamy relations in the multipartite system. As Ref. [28] has shown analytically the following monogamy relation

$$\mathcal{C}_{q_1|(q_2\cdots q_n)}^2 \geq \mathcal{C}_{q_1|q_2}^2 + \mathcal{C}_{q_1|q_3}^2 + \cdots + \mathcal{C}_{q_1|q_n}^2 \qquad (1.12)$$

holds in the n-qubit pure-state system. However, the leftover of Eq. (1.12) is not entan-glement monotone. The authors in Ref. [29, 30] conjectured that in four-qubit system the following quantity

$$t_1 = \frac{\pi_A + \pi_B + \pi_C + \pi_D}{4} \qquad (1.13)$$

is a monotone, where $\pi_A = \mathcal{C}_{A|(BCD)}^2 - (\mathcal{C}_{A|B}^2 + \mathcal{C}_{A|C}^2 + \mathcal{C}_{A|D}^2)$ and other ones are obtained by changing the focusing qubit. Even though t_1 might be an entanglement monotone, it is obvious that it is not a true four-way measure because it detects the three-way entanglement. For example, $t_1(g_3) = 3/4$, where $|g_3\rangle = (|0000\rangle + |1110\rangle)/\sqrt{2}$.

In Ref. [31] another following multipartite monogamy relation is derived:

$$
\mathcal{C}_{q_1|(q_2\cdots q_n)}^2 \geq \underbrace{\sum_{j=2}^{n} \mathcal{C}_{q_1|q_j}^2}_{2-\text{partite}} + \underbrace{\sum_{k>j=2}^{n} \left[\tau_{q_1|q_j|q_k}\right]^{\mu_3}}_{3-\text{partite}} + \cdots + \underbrace{\sum_{\ell=2}^{n} \left[\tau_{q_1|q_2|\cdots|q_{\ell-1}|q_{\ell+1}|\cdots|q_n}\right]^{\mu_{n-1}}}_{(n-1)-\text{partite}}. \tag{1.14}
$$

In Eq. (1.14) the power factors $\{\mu_m\}_{m=3}^{n-1}$ are included to regulate the weight assigned to the different m-partite contributions. If all power factors μ_m go to infinity, Eq. (1.14) reduces to Eq. (1.12). Especially, the authors in Ref. [31] have conjectured $\mu_3 = 3/2$[5]. Thus, in four-qubit system one can construct another possible candidate of the tangle-based entanglement measure

$$
t_2 = \frac{\sigma_A + \sigma_B + \sigma_C + \sigma_D}{4}, \tag{1.15}
$$

where $\sigma_A = \mathcal{C}_{A|(BCD)}^2 - \left(\mathcal{C}_{A|B}^2 + \mathcal{C}_{A|C}^2 + \mathcal{C}_{A|D}^2\right) - \left(\left[\tau_{A|B|C}\right]^{\mu} + \left[\tau_{A|B|D}\right]^{\mu} + \left[\tau_{A|C|D}\right]^{\mu}\right)$, and others are obtained by changing the focusing qubit. One can show easily $t_2(g_3) = 0$. Thus, the measure t_2 cannot be excluded as a true four-way entanglement measure.

In Ref. [32, 33] two different negativity-based monogamy relations have been examined. From these relations one can construct the following candidates of the four-party entanglement measures:

$$
n_1 = \frac{u_A + u_B + u_C + u_D}{4} \tag{1.16}
$$

where $u_A = \mathcal{N}_{A|(BCD)} - \left(\mathcal{N}_{A|B} + \mathcal{N}_{A|C} + \mathcal{N}_{A|D}\right) - \left(\left[\mathcal{N}_{A||B|C}\right]^{\nu_1} + \left[\mathcal{N}_{A||B|D}\right]^{\nu_1} + \left[\mathcal{N}_{A||C|D}\right]^{\nu_1}\right)$ with $\mathcal{N}_{I||J|K} \equiv \mathcal{N}_{I|(JK)} - \mathcal{N}_{I|J} - \mathcal{N}_{I|K}$ and

$$
n_2 = \frac{v_A + v_B + v_C + v_D}{4} \tag{1.17}
$$

where $v_A = \mathcal{N}_{A|(BCD)}^2 - \left(\mathcal{N}_{A|B}^2 + \mathcal{N}_{A|C}^2 + \mathcal{N}_{A|D}^2\right) - \left(\left[\mathcal{N}_{A||B|C}^2\right]^{\nu_2} + \left[\mathcal{N}_{A||B|D}^2\right]^{\nu_2} + \left[\mathcal{N}_{A||C|D}^2\right]^{\nu_2}\right)$ with $\mathcal{N}_{I||J|K}^2 \equiv \mathcal{N}_{I|(JK)}^2 - \mathcal{N}_{I|J}^2 - \mathcal{N}_{I|K}^2$. The negativity $\mathcal{N}$ is defined as [34, 35]

$$
\mathcal{N}(\rho_{AB}) = ||\rho_{AB}^{T_A}|| - 1 \tag{1.18}
$$

where $||X|| \equiv \text{tr}(\sqrt{XX^\dagger})$ and the superscript T_A means the partial transposition of A-qubit. Of course other quantities can be obtained by changing the focusing qubit.

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[5] In fact, this factor is unclear because it can be cubic power of the three tangle or three a half power of the residual entanglement.

The purpose of this paper is to test t_1, t_2, n_1, and n_2 by computing them for the rank-2 mixture

$$\rho_4 = p|\text{GHZ}_4\rangle\langle\text{GHZ}_4| + (1-p)|\text{W}_4\rangle\langle\text{W}_4| \qquad (1.19)$$

where $|\text{GHZ}_4\rangle$ is defined in Eq. (1.10) and $|\text{W}_4\rangle = (\sigma_x \otimes \sigma_x \otimes \sigma_x \otimes \sigma_x)|\tilde{\text{W}}_4\rangle$. In section II we compute t_1, t_2, n_1, and n_2 for the maximal entangled pure states (1.10) and $|\text{W}_4\rangle$. The results are summarized in Table II. It is shown that the negativity-based measures n_1 and n_2 become negative for $|\Phi_3\rangle$. In section III we try to compute t_1 and t_2 for ρ_4 by finding the optimal decompositions. For t_1 it turns out that Eq. (1.19) itself is an optimal decomposition. However, we fail to compute t_2 because analytic computation of the residual entanglement is extremely difficult. In section IV we compute n_1 and n_2 for ρ_4 in the range $\nu_{1*} \leq \nu_1$ and $\nu_{2*} \leq \nu_2$ by finding the optimal decompositions, where

$$\nu_{1*} = \frac{\ln\left(\frac{3-3\sqrt{2}+\sqrt{3}}{6}\right)}{\ln\left(\frac{3}{2}-\sqrt{2}\right)} = 1.02053 \quad \nu_{2*} = \frac{\ln\left(\frac{\sqrt{2}-1}{2}\right)}{\ln\left(\sqrt{2}-\frac{5}{4}\right)} = 0.871544. \qquad (1.20)$$

In this region $n_1(\text{W}_4)$ and $n_2(\text{W}_4)$ become non-negative. In section V a brief conclusion is given. In appendix we try to explain why the computation of the residual entanglement is highly difficult.

II. COMPUTATION OF t_1, t_2, n_1, AND n_2 FOR FEW SPECIAL PURE STATES

In this section we compute t_1, t_2, n_1, and n_2 for four-qubit maximally entangled states (1.10) and W-state $|\text{W}_4\rangle$. The most special case is $|\text{GHZ}_4\rangle$, which gives $t_1 = t_2 = n_1 = n_2 = 1$. Since $|\text{W}_4\rangle$ saturates the monogamy relations (1.12) and (1.14), t_1 and t_2 of $|\text{W}_4\rangle$ are exactly zero. However, $|\text{W}_4\rangle$ does not saturate the negativity-based monogamy relations. It is straightforward[33] to show that n_1 and n_2 of $|\text{W}_4\rangle$ are

$$n_1(\text{W}_4) = \frac{3+\sqrt{3}-3\sqrt{2}}{2} - 3\left(\frac{3-2\sqrt{2}}{2}\right)^{\nu_1} \qquad (2.1)$$

$$n_2(\text{W}_4) = \frac{3}{2}(\sqrt{2}-1) - 3\left(\frac{4\sqrt{2}-5}{4}\right)^{\nu_2}.$$

Thus, as we commented, $n_1(\text{W}_4)$ and $n_2(\text{W}_4)$ become non-negative when $\nu_{1*} \leq \nu_1$ and $\nu_{2*} \leq \nu_2$.

For $|\Phi_2\rangle$ it is easy to show that $\mathcal{C}_{I|(JKL)} = 1$ and $\mathcal{C}_{I|J} = 0$ for all $\{I, J, K, L\} = \{A, B, C, D\}$. The tripartite states derived from $|\Phi_2\rangle\langle\Phi_2|$ by tracing over any one-party is given by

$$\rho_3^{(2)} = \frac{1}{2}|\psi_1\rangle\langle\psi_1| + \frac{1}{2}|W_3\rangle\langle W_3| \tag{2.2}$$

where $|\psi_1\rangle = (|000\rangle + \sqrt{2}|111\rangle)/\sqrt{3}$ and $|W_3\rangle = (|001\rangle + |010\rangle + |100\rangle)/\sqrt{3}$. In order to compute the residual entanglement of $\rho_3^{(2)}$ let us consider the quantum state $p|gGHZ\rangle\langle gGHZ| + (1-p)|gW\rangle\langle gW|$, where $|gGHZ\rangle = a|000\rangle + b|111\rangle$ and $|gW\rangle = c|001\rangle + d|010\rangle + f|100\rangle$. As the second reference of Ref. [21] has shown, the residual entanglement of this state is exactly zero when $p \leq p_0 \equiv s^{2/3}/(1 + s^{2/3})$ where $s = 4cdf/(a^2b)$. Since, for our case, $p_0 = 2/3$ is larger than $1/2$, the residual entanglement of $\rho_3^{(2)}$ is zero. Thus, t_1 and t_2 of $|\Phi_2\rangle$ are

$$t_1(\Phi_2) = t_2(\Phi_2) = 1. \tag{2.3}$$

Various negativities of $|\Phi_2\rangle$ can be directly computed and the final expressions are

$$\mathcal{N}_{I|(JKL)} = 1 \qquad \mathcal{N}_{I|(JK)} = \frac{2}{3} \qquad \mathcal{N}_{I|J} = 0 \tag{2.4}$$

for all $\{I, J, K, L\} = \{A, B, C, D\}$. Thus, it is easy to show

$$n_1(\Phi_2) = 1 - 3\left(\frac{2}{3}\right)^{\nu_1} \qquad n_2(\Phi_2) = 1 - 3\left(\frac{4}{9}\right)^{\nu_2}. \tag{2.5}$$

For $|\Phi_3\rangle$ one can show $\mathcal{C}_{I|(JKL)} = 1$ and $\mathcal{C}_{I|J} = 0$ for all $\{I, J, K, L\}$. The tripartite states derived from $|\Phi_3\rangle\langle\Phi_3|$ are

$$\rho_{ACD}^{(3)} = \rho_{BCD}^{(3)} = \frac{1}{2}|\phi_1\rangle\langle\phi_1| + \frac{1}{2}|\phi_2\rangle\langle\phi_2| \tag{2.6}$$

$$\rho_{ABC}^{(3)} = \rho_{ABD}^{(3)} = \frac{1}{2}|g_1\rangle\langle g_1| + \frac{1}{2}|g_2\rangle\langle g_2|$$

where

$$|\phi_1\rangle = \frac{1}{\sqrt{2}}(|100\rangle + |111\rangle) \qquad |\phi_2\rangle = \frac{1}{\sqrt{2}}(|001\rangle + |010\rangle) \tag{2.7}$$

$$|g_1\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) \qquad |g_2\rangle = \frac{1}{\sqrt{2}}(|001\rangle + |110\rangle).$$

It is easy to show that the residual entanglements of $\rho_{ACD}^{(3)}$ and $\rho_{BCD}^{(3)}$ are zero because $|\phi_1\rangle$ and $|\phi_2\rangle$ are bi-separable. In order to compute the residual entanglement of $\rho_{ABC}^{(3)}$ and $\rho_{ABD}^{(3)}$ let us consider the quantum state $p|g_1\rangle\langle g_1| + (1-p)|g_2\rangle\langle g_2|$. As the last reference of Ref.

[21] has shown, the residual entanglement of this state is $(2p - 1)^2$. Thus, the residual entanglements of $\rho_{ABC}^{(3)}$ and $\rho_{ABD}^{(3)}$ are also zero, all of which yields

$$t_1(\Phi_3) = t_2(\Phi_3) = 1. \tag{2.8}$$

Various negativities of $|\Phi_3\rangle$ can be computed directly and the final expressions are

$$\mathcal{N}_{I|(JKL)} = 1 \qquad \mathcal{N}_{I|J} = 0 \tag{2.9}$$

for all $\{I, J, K, L\} = \{A, B, C, D\}$ and

$$\mathcal{N}_{A|(CD)} = \mathcal{N}_{B|(CD)} = \mathcal{N}_{C|(AB)} = \mathcal{N}_{D|(AB)} = 0 \tag{2.10}$$

$$\mathcal{N}_{A|(BC)} = \mathcal{N}_{A|(BD)} = \mathcal{N}_{B|(AC)} = \mathcal{N}_{B|(AD)} = \mathcal{N}_{C|(AD)} = \mathcal{N}_{C|(BD)} = \mathcal{N}_{D|(AC)} = \mathcal{N}_{D|(BC)} = 1,$$

all of which yields

$$n_1(\Phi_3) = n_2(\Phi_3) = -1. \tag{2.11}$$

All results are summarized in Table II.

	t_1	t_2	n_1	n_2
$|\mathrm{GHZ}_4\rangle$	1	1	1	1
$|\Phi_2\rangle$	1	1	$1 - 3\left(\frac{2}{3}\right)^{\nu_1}$	$1 - 3\left(\frac{4}{9}\right)^{\nu_2}$
$|\Phi_3\rangle$	1	1	-1	-1
$|\mathrm{W}_4\rangle$	0	0	$\frac{3+\sqrt{3}-3\sqrt{2}}{2} - 3\left(\frac{3-2\sqrt{2}}{2}\right)^{\nu_1}$	$\frac{3}{2}(\sqrt{2}-1) - 3\left(\frac{4\sqrt{2}-5}{4}\right)^{\nu_2}$

Table II: t_1, t_2, n_1, and n_2 of the maximally entangled and W_4 states.

III. TANGLE-BASED ENTANGLEMENT MEASURES FOR RANK-2 MIXTURE

In this section we try to compute t_1 and t_2 for the rank-2 mixture ρ_4. For computation of t_1 and t_2 we have to find an optimal decomposition. In order to find the optimal decompositions for t_1 and t_2 we define

$$|Z_4(p, \varphi)\rangle = \sqrt{p}|\mathrm{GHZ}_4\rangle - e^{i\varphi}\sqrt{1 - p}|\mathrm{W}_4\rangle. \tag{3.1}$$

Then, one can show that any reduced bipartite states from $|Z_4(p, \varphi)\rangle\langle Z_4(p, \varphi)|$ are all equal to

$$\rho_{IJ} = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{\frac{p(1-p)}{2}}e^{-i\varphi} & -\sqrt{\frac{p(1-p)}{2}}e^{-i\varphi} & 0 \\ -\sqrt{\frac{p(1-p)}{2}}e^{i\varphi} & \frac{1-p}{2} & \frac{1-p}{2} & 0 \\ -\sqrt{\frac{p(1-p)}{2}}e^{i\varphi} & \frac{1-p}{2} & \frac{1-p}{2} & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \quad (I, J = A, B, C, D) \quad (3.2)$$

in the computational basis. Although ρ_{IJ} is a rank-3 mixture, one can compute its concurrence analytically by following Wootters procedure:

$$\mathcal{C}_{I|J} = \sqrt{\Lambda} - \sqrt{\Lambda_+} - \sqrt{\Lambda_-} \quad (3.3)$$

where

$$\Lambda = \frac{1}{12}\left[(1 + p^2) + 2(\alpha^2 + \beta^2)^{1/6} \cos\theta\right] \quad (3.4)$$

$$\Lambda_\pm = \frac{1}{12}\left[(1 + p^2) - 2(\alpha^2 + \beta^2)^{1/6} \cos\left(\frac{\pi}{3} \pm \theta\right)\right]$$

and

$$\alpha = 1 - 9p + 39p^2 - 90p^3 + \frac{231}{2}p^4 - 81p^5 + \frac{47}{2}p^6$$

$$\beta = \frac{3p^2(1-p)}{2}\sqrt{3p(4 - 28p + 96p^2 - 147p^3 + 110p^4 - 31p^5)} \quad (3.5)$$

$$\theta = \frac{1}{3}\tan^{-1}\left(\frac{\beta}{\alpha}\right).$$

It is interesting to note that $\mathcal{C}_{I|J} = 0$ at $p = 1/3$. Furthermore, it is worthwhile noting that $\mathcal{C}_{I|J}$ is independent of the phase factor φ. On the contrary, the corresponding concurrence $\mathcal{C}_{I|J}^{(3)}$ derived from the three-qubit state $|Z_3(p, \varphi)\rangle = \sqrt{p}|\text{GHZ}_3\rangle - e^{i\varphi}\sqrt{1-p}|W_3\rangle$ is explicitly dependent on φ. The p-dependence of $\mathcal{C}_{I|J}$ and $\mathcal{C}_{I|J}^{(3)}$ are plotted in Fig. 1. The φ-dependence of $\mathcal{C}_{I|J}^{(3)}$ makes the three-qubit rank-2 mixture $\rho_3 = p|\text{GHZ}_3\rangle\langle\text{GHZ}_3| + (1-p)|W_3\rangle\langle W_3|$ have the nontrivial residual entanglement[21]. As we will show shortly, the φ-independence of $\mathcal{C}_{I|J}$ makes t_1 of ρ_4 to be trivial.

The single qubit states derived from $|Z_4(p, \varphi)\rangle\langle Z_4(p, \varphi)|$ are all equal to

$$\rho_J = \begin{pmatrix} \frac{3-p}{4} & -\frac{1}{2}\sqrt{\frac{p(1-p)}{2}}e^{-i\varphi} \\ -\frac{1}{2}\sqrt{\frac{p(1-p)}{2}}e^{i\varphi} & \frac{1+p}{4} \end{pmatrix} \quad (J = A, B, C, D) \quad (3.6)$$
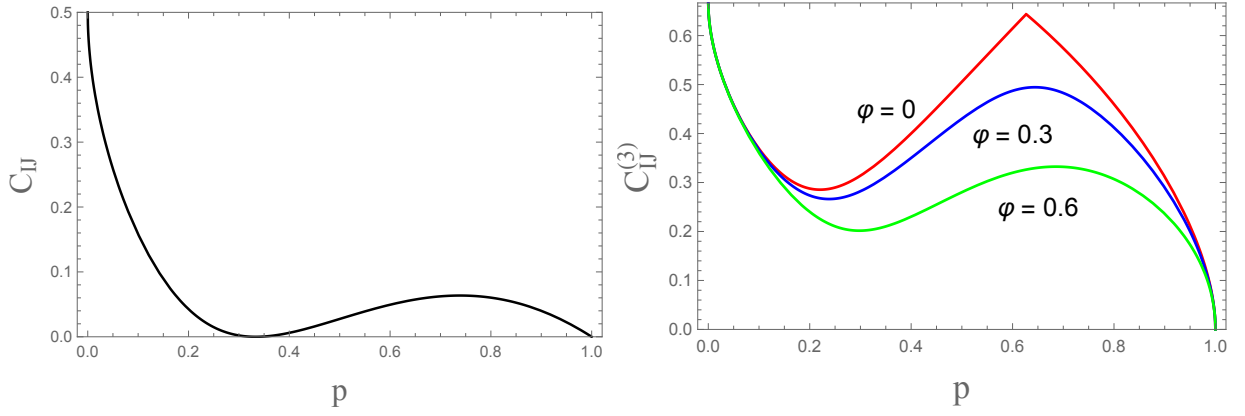
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FIG. 1: (Color online) The p-dependence of (a) $\mathcal{C}_{I|J}$ and (b) $\mathcal{C}_{I|J}^{(3)}$. As Fig. (a) shows, $\mathcal{C}_{I|J}$ is independent of the phase factor φ unlike $\mathcal{C}_{I|J}^{(3)}$. This makes $t_1(\rho_4)$ trivial.

in the computational basis. Thus, $\mathcal{C}_{I|(JKL)}^2$ for $|Z_4(p,\varphi)\rangle$ is

$$\mathcal{C}_{I|(JKL)}^2 = 4\det\rho_I = \frac{3+p^2}{4} \tag{3.7}$$

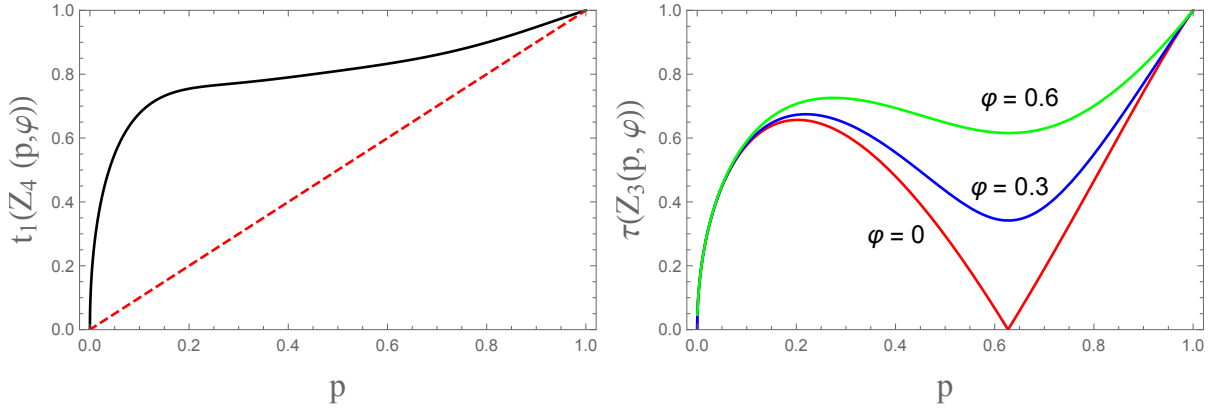for all I, J, K. The corresponding results derived from $|Z_3(p,\varphi)\rangle$ is $(8 - 4p + 5p^2)/9$.

FIG. 2: (Color online) The p-dependence of (a) $t_1\left(Z_4(p,\varphi)\right)$ and (b) $\tau\left(Z_3(p,\varphi)\right)$. On the contrary to $\tau\left(Z_3(p,\varphi)\right)$ $t_1\left(Z_4(p,\varphi)\right)$ is independent of the phase factor φ.

Thus, $t_1\left(Z_4(p,\varphi)\right)$ is independent of φ as

$$t_1\left(Z_4(p,\varphi)\right) = \frac{3+p^2}{4} - 3\mathcal{C}_{I|J}^2. \tag{3.8}$$

The corresponding residual entanglement $\tau\left(Z_3(p,\varphi)\right)$ for $|Z_3(p,\varphi)\rangle$ is dependent on φ due to $\mathcal{C}_{I|J}^{(3)}$. The p-dependence of $t_1\left(Z_4(p,\varphi)\right)$ and $\tau\left(Z_3(p,\varphi)\right)$ is plotted in Fig. 2(a) and Fig. 2(b) respectively.

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Before we calculate $t_1(\rho_4)$ it seems to be helpful to review briefly how to compute $\tau(\rho_3)$ for $\rho_3 = p|\text{GHZ}_3\rangle\langle\text{GHZ}_3| + (1-p)|\text{W}_3\rangle\langle\text{W}_3|$. As Fig. 2(b) shows, when $\varphi = 0$ $\tau(Z_3(p,\varphi))$ has nontrivial zero at $p = p_0$ with $p_0 \sim 0.627$. Furthermore, $\tau(Z_3(p,\varphi = 0))$ is not convex in the regions $0 \leq p \leq p_0$ and $p_1 \leq p \leq 1$ with $p_1 \sim 0.826$. Since $\tau(Z_3(p,\varphi))$ depends on φ through only $\cos 3\varphi$, $\tau(Z_3(p_0, \varphi)) = 0$ for $\varphi = 0, 2\pi/3, 4\pi/3$. Thus, at the small concave region it is possible to convexify the residual entanglement by making use of $\{|\text{W}_3\rangle, |Z_3(p, \frac{2\pi}{3}j)\rangle$ $(j = 0, 1, 2)\}$. At the large concave region it is also possible to convexify it by making use of $\{|\text{GHZ}_3\rangle, |Z_3(p, \frac{2\pi}{3}j)\rangle$ $(j = 0, 1, 2)\}$.

Now, let us return to the four-qubit case. As Fig. 2(a) shows $t_1(Z_4(p,\varphi))$ is not convex at $0 \leq p \leq p_0$ and $p_1 \leq p \leq 1$ with $p_0 \sim 0.279$ and $p_1 \sim 0.936$. As the three-qubit case it is possible to convexify the entanglement by making use of $\{|\text{GHZ}_4\rangle, |Z_4(p,0)\rangle, |Z_4(p,\pi)\rangle\}$ in the large p-region. However, it is impossible to convexify it in the small p-region because $t_1(Z_4(p_0, 0)) \neq 0$. The only way to obtain the convex result in the entire range of p is

$$t_1(\rho_4) = p. \tag{3.9}$$

As Fig. 2(a) shows obviously as a dashed line this is a convex hull of $t_1(Z_4(p,\varphi))$. Thus the optimal decomposition for t_1 is nothing but the spectral decomposition (1.19) itself.

In order to compute $t_2(\rho_4)$ we should compute the residual entanglement for the three-qubit states reduced from $|Z_4(p,\varphi)\rangle$. One can show that all tripartite states derived by tracing over single qubit are equal to

$$\rho_{IJK} = \lambda|\psi_+\rangle\langle\psi_+| + (1-\lambda)|\psi_-\rangle\langle\psi_-| \quad (I, J, K = A, B, C, D) \tag{3.10}$$

where

$$\lambda = \frac{2 + \sqrt{1-p^2}}{4} \tag{3.11}$$

$$|\psi_\pm\rangle = \frac{1}{N_\pm}\left[\mu_\pm|000\rangle - e^{-i\varphi}|111\rangle - \nu_\pm e^{i\varphi}\left(|001\rangle + |010\rangle + |100\rangle\right)\right]$$

with

$$N_\pm^2 = \frac{(1+p)(3-p) \pm (3+p)\sqrt{1-p^2}}{2p^2} \qquad \mu_\pm = \frac{2(1-p) \pm \sqrt{1-p^2}}{\sqrt{2p(1-p)}} \tag{3.12}$$

$$\nu_\pm = \frac{(3+p)(1-p) \pm (3-p)\sqrt{1-p^2}}{2p(2 \pm \sqrt{1-p^2})}.$$

The residual entanglement for $|\psi_\pm\rangle$ is

$$\tau(\psi_\pm) = \frac{4}{N_\pm^4}\sqrt{\mu_\pm^4 + 16\nu_\pm^6 + 8\mu_\pm^2\nu_\pm^3\cos 4\varphi}. \tag{3.13}$$

Thus, the spectral decomposition (3.10) indicates that the residual entanglement for ρ_{IJK} satisfies

$$\tau(\rho_{IJK}) \leq \lambda\tau(\psi_+) + (1-\lambda)\tau(\psi_-). \tag{3.14}$$

However, the analytic computation of the residual entanglement for ρ_{IJK} is highly difficult even though it is rank-2 tensor. In appendix we try to describe why it is highly difficult. Therefore, we fail to compute $t_2(\rho_4)$ analytically.

IV. NEGATIVITY-BASED ENTANGLEMENT MEASURES FOR RANK-2 MIXTURE

In this section we try to compute n_1 and n_2 for ρ_4. We consider only the regions $\nu_{1*} \leq \nu_1 \leq \infty$ and $\nu_{2*} \leq \nu_2 \leq \infty$, where ν_{1*} and ν_{2*} are given in Eq. (1.20). When $\nu_1 = \nu_{1*}$ and $\nu_2 = \nu_{2*}$, $n_1(W_4) = n_2(W_4) = 0$ exactly. When $\nu_1 = \nu_2 = \infty$, n_1 and n_2 for $|W_4\rangle$ become

$$n_1(W_4) = \frac{3 + \sqrt{3} - 3\sqrt{2}}{2} = 0.244705 \quad n_2(W_4) = \frac{3}{2}(\sqrt{2} - 1) = 0.62132. \tag{4.1}$$

Of course, n_1 and n_2 for $|GHZ_4\rangle$ are unity regardless of ν_1 and ν_2.

In order to find the optimal decompositions for $n_1(\rho_4)$ and $n_2(\rho_4)$ we re-consider $|Z_4(p,\varphi)\rangle$ defined in Eq. (3.1). By direct calculation one can show straightforwardly

$$\mathcal{N}_{I|(JKL)} = \frac{1}{2}\sqrt{3 + p^2} \tag{4.2}$$

where I, J, K, L are any one of $\{A, B, C, D\}$.

Using Eq. (3.2) one can also show that any bipartite negativity $\mathcal{N}_{I|J}$ for $|Z_4(p,\varphi)\rangle$ is

$$\mathcal{N}_{I|J} = \sqrt{\lambda} + \sqrt{\lambda_+} + \sqrt{\lambda_-} - \frac{3+p}{4} \tag{4.3}$$

where

$$\lambda = \frac{1}{48}\left[(7 + 2p - p^2) + 4r_0\cos\theta_0\right] \quad \lambda_\pm = \frac{1}{48}\left[(7 + 2p - p^2) - 4r_0\cos\left(\frac{\pi}{3} \pm \theta_0\right)\right] \tag{4.4}$$

with

$$r_0 = (\alpha_0^2 + \beta_0^2)^{1/6} \quad \theta_0 = \frac{1}{3}\tan^{-1}\left(\frac{\beta_0}{\alpha_0}\right)$$
$$\alpha_0 = 17 + 147p - 153p^2 - 428p^3 + 729p^4 - 447p^5 + 127p^6 \tag{4.5}$$
$$\beta_0 = 3\sqrt{3}(1 - p + 5p^2 - 7p^3 + 2p^4)\sqrt{2 + 4p - 71p^2 + 214p^3 - 129p^4}.$$

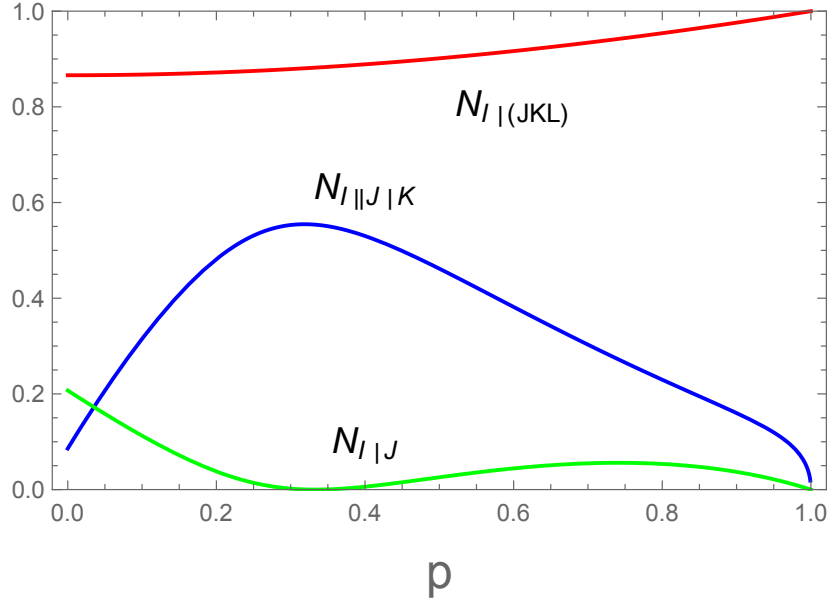Like the concurrence discussed in the previous section $\mathcal{N}_{I|J}$ becomes zero when $p = 1/3$ and $p = 1$.

FIG. 3: (Color online) The p-dependence of $\mathcal{N}_{I|(JKL)}$, $\mathcal{N}_{I||J|K}$, and $\mathcal{N}_{I|J}$ for $|Z(p,\varphi)\rangle$. It is shown that all negativities are independent of the phase factor φ.

Finally, we compute $\mathcal{N}_{I||(JK)}$ for all $I, J, K = A, B, C, D$. Using Eq. (3.10) one can compute the non-zero eigenvalues of $\left(\rho_{IJK}^{T_I}\right)\left(\rho_{IJK}^{T_I}\right)^\dagger$. One of them is $p^2/4$ and the remaining five non-zero eigenvalues can be obtained by solving the quintic equation. Thus, it is possible to compute $\mathcal{N}_{I||(JK)}$ numerically. After obtaining $\mathcal{N}_{I||(JK)}$, one can compute $\mathcal{N}_{I||J|K}$ and $\mathcal{N}_{I||J|K}^2$ by making use of $\mathcal{N}_{I||J|K} = \mathcal{N}_{I||(JK)} - (\mathcal{N}_{I|J} + \mathcal{N}_{I|K})$ and $\mathcal{N}_{I||J|K}^2 = \mathcal{N}_{I||(JK)}^2 - (\mathcal{N}_{I|J}^2 + \mathcal{N}_{I|K}^2)$. It is worthwhile noting that all negativities are independent of the phase angle φ. Thus, $n_1(\rho_4)$ and $n_2(\rho_4)$ are independent of φ. In Fig. 3 we plot the p-dependence of $\mathcal{N}_{I|(JKL)}$, $\mathcal{N}_{I||J|K}$, and $\mathcal{N}_{I|J}$.

In Fig. 4(a) we plot the p-dependence of $n_1(Z(p,\varphi))$ for $|Z_4(p,\varphi)\rangle$ when $\nu_1 = \nu_{1*}$ (red dashed line) and $\nu_1 = \infty$ (blue dashed line). When $\nu_1 = \nu_{1*}$, $n_1(Z(p,\varphi))$ becomes negative

at $0 < p < p_0$, where $p_0 = 0.749596$. Since, however, $n_1(W_4) = 0$ at $\nu_1 = \nu_{1*}$, one can choose the optimal decomposition for $\rho_4(p)$ in this region as

$$\rho_4(p) = \frac{p}{2p_0}\left[|Z_4(p_0, 0)\rangle\langle Z_4(p_0, 0)| + |Z_4(p_0, \pi)\rangle\langle Z_4(p_0, \pi)|\right] + \left(1 - \frac{p}{p_0}\right)|W_4\rangle\langle W_4|, \quad (4.6)$$

which gives $n_1(\rho_4) = 0$ at $0 \le p \le p_0$. At $p_0 \le p \le 1$ the optimal decomposition for $\rho_4(p)$ is

$$\rho_4(p) = \frac{1}{2}\left[|Z_4(p, 0)\rangle\langle Z_4(p, 0)| + |Z_4(p, \pi)\rangle\langle Z_4(p, \pi)|\right], \quad (4.7)$$

which gives $n_1(\rho_4) = n_1(Z(p, \varphi))$ at $p_0 \le p \le 1$. Since $n_1(Z(p, \varphi))$ is convex in this region, we do not need to convexify it. Thus, our result for $n_1(\rho_4)$ at $\nu_1 = \nu_{1*}$ can be expressed as

$$n_1(\rho_4) = \begin{cases} 0 & 0 \le p \le p_0 = 0.749596 \\ n_1(Z(p, 0)) & p_0 \le p \le 1. \end{cases} \quad (4.8)$$

This is plotted in Fig. 4(a) as a red solid line.

When $\nu_1 = \infty$, $n_1(Z(p, \varphi))$ is not convex at the region $0 \le p \le p_*$ with $p_* \approx 0.475$. Thus, we have to convexify it at the region $0 \le p \le p_1$ with $p_1 > p_*$. We will fix p_1 later. At the region $0 \le p \le p_1$ we choose an optimal decomposition for $\rho_4(p)$ as

$$\rho_4(p) = \frac{p_1 - p}{p_1}|W_4\rangle\langle W_4| + \frac{p}{2p_1}\left[|Z_4(p_1, 0)\rangle\langle Z_4(p_1, 0)| + |Z_4(p_1, \pi)\rangle\langle Z_4(p_1, \pi)|\right]. \quad (4.9)$$

From Eq. (4.9) $n_1(\rho_4)$ becomes $g(p)$, where

$$g(p) = \frac{3 + \sqrt{3} - 3\sqrt{2}}{2}\frac{p_1 - p}{p_1} + \frac{p}{p_1}n_1(Z(p_1, 0)). \quad (4.10)$$

Then, p_1 is determined by $\partial g(p, p_1)/\partial p_1 = 0$, which gives $p_1 \approx 0.84$. Thus, finally $n_1(\rho_4)$ at $\nu_1 = \infty$ is given by

$$n_1(\rho_4) = \begin{cases} g(p) & 0 \le p \le p_1 \approx 0.84 \\ n_1(Z(p, 0)) & p_1 \le p \le 1. \end{cases} \quad (4.11)$$

This is plotted in Fig. 4(a) as a blue solid line.

In Fig. 4(b) we plot the p-dependence of $n_2(Z(p, \varphi))$ for $|Z_4(p, \varphi)\rangle$ when $\nu_2 = \nu_{2*}$ (red dashed line) and $\nu_2 = \infty$ (blue dashed line). When $\nu_2 = \nu_{2*}$, $n_2(Z(p, \varphi))$ becomes negative at the region $0 \le p \le p_0$, where $p_0 = 0.57731$. Following the similar procedure in the case of $\nu_1 = \nu_{1*}$ one can derive $n_2(\rho_4)$ as

$$n_2(\rho_4) = \begin{cases} 0 & 0 \le p \le p_0 = 0.57731 \\ n_2(Z(p, 0)) & p_0 \le p \le 1. \end{cases} \quad (4.12)$$
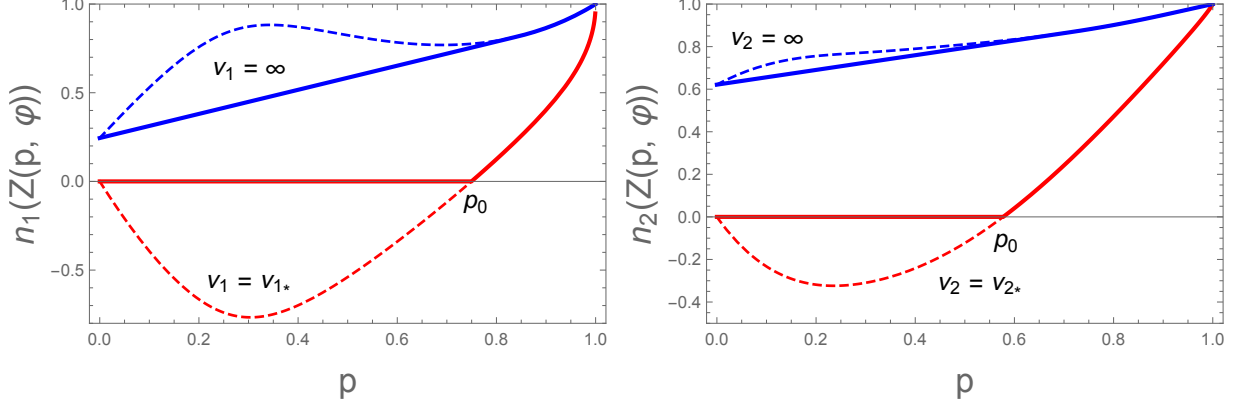
FIG. 4: (Color online) The p-dependence of (a) $n_1(Z(p,\varphi))$ (dashed) and $n_1(\rho_4)$ (solid) at $\nu_1 = \nu_{1*}$ (red) and $\nu_1 = \infty$ (blue) (b) $n_2(Z(p,\varphi))$ (dashed) and $n_2(\rho_4)$ (solid) at $\nu_2 = \nu_{2*}$ (red) and $\nu_2 = \infty$ (blue).

This is plotted in Fig. 4(b) as a red solid line.

For $\nu_2 = \infty$ case $n_2(Z(p,\varphi))$ is not convex at $0 \leq p \leq p_{1*}$ and $p_{2*} \leq p \leq 1$, where $p_{1*} \approx 0.25$ and $p_{2*} \approx 0.95$. Thus, we have to convexify $n_2(\rho_4)$ in the small-p and large-p regions. First we choose a small-p region $0 \leq p \leq p_1$ with $p_{1*} \leq p_1 \leq p_{2*}$. The parameter p_1 will be fixed later. In this region we choose the optimal decomposition as Eq. (4.9). Then, $n_2(\rho_4)$ becomes $f_I(p)$, where

$$f_I(p) = \frac{3}{2}(\sqrt{2} - 1)\frac{p_1 - p}{p_1} + \frac{p}{p_1}n_2(Z(p_1, 0)). \tag{4.13}$$

Then, p_1 is fixed by $\partial f_I(p, p_1)/\partial p_1 = 0$, which gives $p_1 \approx 0.72$. Next, we consider the large-p region $p_2 \leq p \leq 1$ with $p_1 \leq p_2 \leq p_{2*}$. In this region the optimal decomposition can be chosen as

$$\rho_4(p) = \frac{p - p_2}{1 - p_2}|\text{GHZ}_4\rangle\langle\text{GHZ}_4| + \frac{1 - p}{2(1 - p_2)}\left[|Z_4(p_2, 0)\rangle\langle Z_4(p_2, 0)| + |Z_4(p_2, \pi)\rangle\langle Z_4(p_2, \pi)|\right]. \tag{4.14}$$

Thus, $n_2(\rho_4)$ becomes $f_{II}(p)$ in this region, where

$$f_{II}(p) = \frac{p - p_2}{1 - p_2} + \frac{1 - p}{1 - p_2}n_2(Z(p_2, 0)). \tag{4.15}$$

The parameter p_2 is fixed by $\partial f_{II}(p, p_2)/\partial p_2 = 0$, which gives $p_2 \approx 0.92$. Thus, the final

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expression $n_2(\rho_4)$ for $\nu_2 = \infty$ case can be written in a form

$$
n_2(\rho_4) = \begin{cases} f_I(p) & 0 \leq p \leq p_1 \approx 0.72 \\ n_2(Z(p,0)) & p_1 \leq p \leq p_2 \approx 0.92 \\ f_{II}(p) & p_2 \leq p \leq 1. \end{cases} \tag{4.16}
$$

This is plotted as a blue solid line in Fig. 4(b).

V. CONCLUSIONS

In this paper we compute the monogamy-motivated four-party measures n_1, n_2, t_1, and t_2 for the rank-2 mixtures ρ_4 given in Eq. (1.19). It turns out that $n_1(\rho_4)$ is trivial and the corresponding optimal decomposition is equal to the spectral decomposition. Probably, this triviality is a sign of the fact that monogamy relation (1.12) is not sufficiently tight, which means that n_1 is not a true four-way entanglement measure. We fail to compute $n_2(\rho_4)$ analytically because it is highly difficult to compute the residual entanglement for the rank-2 state (3.10), which is a tripartite state reduced from $|Z_4(p,\varphi)\rangle\langle Z_4(p,\varphi)|$. This difficulty is discussed in the appendix.

We also compute n_1 and n_2 for ρ_4 when $\nu_j = \nu_{j*}$ $(j = 1, 2)$ or ∞. When $\nu_1 = \nu_{1*}$ the final expression of $n_1(\rho_4)$ is Eq. (4.8) and the corresponding optimal decompositions are (4.6) in $0 \leq p \leq p_0$ and (4.7) in $p_0 \leq p \leq 1$, where $p_0 \sim 0.749596$. When $\nu_1 = \infty$, the final expression of $n_1(\rho_4)$ is Eq. (4.11) and the corresponding optimal decompositions are (4.9) in $0 \leq p \leq p_1$ and (4.7) in $p_1 \leq p \leq 1$, where $p_1 \sim 0.84$. When $\nu_2 = \nu_{2*}$ and $\nu_2 = \infty$, the final expressions of $n_2(\rho_4)$ are Eq. (4.12) and Eq. (4.16), respectively and the corresponding optimal decompositions can be found in the previous section. As Table II shows n_1 and n_2 are not always non-negative for all four-qubit pure states. This means that the corresponding negativity-based monogamy relations discussed in Ref.[32, 33] do not always hold regardless of the power factor ν_1 and ν_2.

It is most important for us to check whether or not t_2 is a true four-way entanglement measure when the power factor μ_3 is chosen appropriately. As we mentioned we fail to check this fact in this paper due to the difficulty for the analytic computation of the residual entanglement of the tripartite reduced state (3.10). We hope to discuss this issue again in the near future.

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Appendix A

In this appendix we try to explain why the analytic computation of the residual entanglement for ρ_{IJK} in Eq. (3.10) is difficult by introducing a simpler rank-2 quantum state

$$\Pi = p|\psi_1\rangle\langle\psi_1| + (1-p)|\psi_2\rangle\langle\psi_2| \tag{A.1}$$

where

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}\left(|\mathrm{GHZ}_3\rangle + |\mathrm{W}_3\rangle\right) \qquad |\psi_2\rangle = \frac{1}{\sqrt{2}}\left(|\mathrm{GHZ}_3\rangle - |\mathrm{W}_3\rangle\right). \tag{A.2}$$

In spite of its simpleness Π has a same structure with ρ_{IJK}. The residual entanglement of $|\psi_1\rangle$ and $|\psi_2\rangle$ are

$$\tau_3(\psi_1) = \frac{8\sqrt{6}+9}{36} = 0.794331 \qquad \tau_3(\psi_2) = \frac{8\sqrt{6}-9}{36} = 0.294331. \tag{A.3}$$

Thus, $\tau_3(\Pi)$ has an upper bound as

$$\tau_3(\Pi) \leq \tau_3^{\max} = \frac{p}{2} + \frac{8\sqrt{6}-9}{36}. \tag{A.4}$$

In order to compute $\tau_3(\Pi)$ we define the superposed state

$$|Z(p,\varphi)\rangle = \sqrt{p}|\psi_1\rangle - e^{i\varphi}\sqrt{1-p}|\psi_2\rangle. \tag{A.5}$$

The residual entanglement $\tau_3(p,\varphi)$ of $|Z(p,\varphi)\rangle$ can be written as

$$\tau_3(p,\varphi) = 4p^2 \left|\frac{1-z}{2}\right| \left|\frac{1}{8}(1-z)^3 + \frac{2}{3\sqrt{6}}(1+z)^3\right| \tag{A.6}$$

where

$$z = e^{i\varphi}\sqrt{\frac{1-p}{p}}. \tag{A.7}$$

Another useful expression of $\tau(p,\varphi)$ is

$$\tau_3(p,\varphi) \tag{A.8}$$
$$= 2\sqrt{\left(1 - 2\sqrt{p(1-p)}\cos\varphi\right)\left(f_0(p) + f_1(p)\cos\varphi + f_2(p)\cos 2\varphi + f_3(p)\cos 3\varphi\right)}$$

where

$$f_0(p) = \frac{155}{1728}(1 + 6p - 6p^2) + \frac{2p-1}{6\sqrt{6}}(1 - 10p + 10p^2)$$
$$f_1(p) = \frac{101}{288}\sqrt{p(1-p)}(1 + p - p^2) \tag{A.9}$$
$$f_2(p) = 6p(1-p)\left(\frac{155}{1728} + \frac{2p-1}{6\sqrt{6}}\right)$$
$$f_3(p) = \frac{101}{864}\sqrt{p^3(1-p)^3}.$$

φ	π	0	$\pi \pm \varphi_0$
p	$p_1 = 0.0163588$	$p_2 = 0.5$	$p_3 = 0.74182$

Table III:Nontrivial zeros of $\tau_3(p, \varphi)$ with $\varphi_0 = 1.27672$.

From Eq. (A.6) one can show that $\tau_3(p, \varphi)$ becomes zero at particular p and φ. This nontrivial zeros are summarized in Table III. From Eq. (A.8) one can show that $\tau_3(p, \varphi)$ has a symmetry

$$\tau_3(p, n\pi + \varphi_0) = \tau_3(p, n\pi - \varphi_0) \tag{A.10}$$

for all integer n.
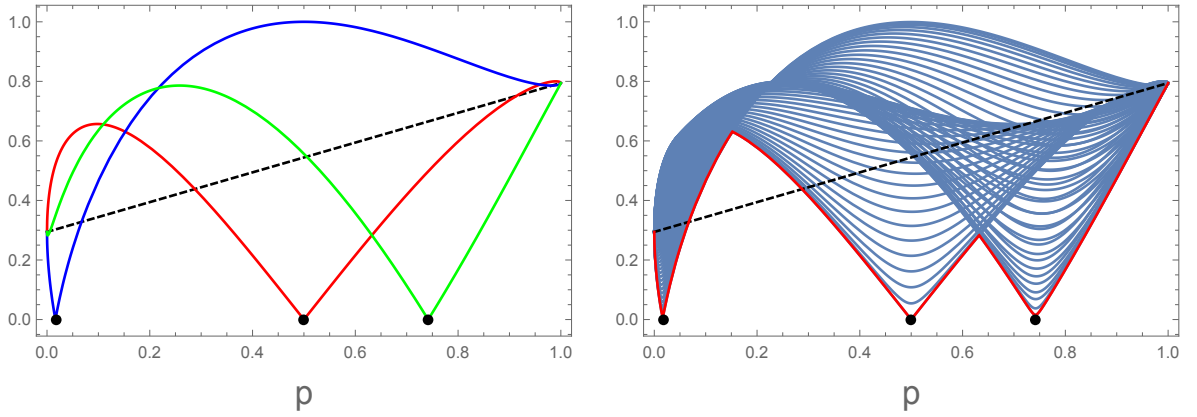

FIG. 5: (Color online) (a) The p-dependence of $\tau_3(p, 0)$, $\tau_3(p, \pi)$, and $\tau_3(p, \pi \pm \varphi_0)$ with $\varphi_0 = 1.27672$. The dashed curve is a p-dependence of $\tau_3^{\max}$. The nontrivial zeros given in Table III are plotted as black dots. (b) The p-dependence of $\tau_3(p, \varphi)$ with varying φ from 0 to π with a step 0.05. These curves have been referred as the characteristic curves. The red solid line is a minimum of the characteristic curves. The red curve seems to indicate that $\tau_3(\Pi)$ is zero at $p_1 \leq p \leq p_3$. However, we cannot find the corresponding optimal decompositions.

In Fig. 5(a) we plot $\tau_3(p, \varphi)$ at $\varphi = 0$, $\pi \pm \varphi_0$, and π with $\varphi_0 = 1.27642$. The nontrivial zeros p_1, p_2, and p_3 given at Table III are plotted as black dots. In Fig. 5(b) we plot $\tau_3(p, \varphi)$ for various φ. These curves have been referred as the characteristic curves. The dashed line in both figures is p-dependence of $\tau_3^{\max}$. The red solid line is a minimum of the characteristic curves. The authors of Ref.[36] have claimed that $\tau_3(\Pi)$ is a convex hull of the minimum of the characteristic curves. If this is right, Fig. 5 (b) seems to exhibit that $\tau_3(\Pi)$ is zero at $p_1 \leq p \leq p_3$. However, it is very difficult to find the corresponding optimal decompositions.

For example, let us consider $p = p_3$ case. Table III indicates that the corresponding optimal decomposition is $(1/2)\left[|Z(p_3, \pi - \varphi_0)\rangle\langle Z(p_3, \pi - \varphi_0)| + |Z(p_3, \pi + \varphi_0)\rangle\langle Z(p_3, \pi + \varphi_0)|\right]$. However, this is not equal to $\Pi(p_3)$ because of the cross terms. Similar difficulties arise at $p = p_1$ and $p = p_2$. So far, we do not know how to compute $\tau_3(\Pi)$ analytically.